20 A Thin, Linear, Vibrating Beam Model: An Example of a Higher Order PDE

For a thin beam of modest displacement, negligible rotary inertia, and stress that can be considered not to vary significantly across a beam section, we have the Euler-Bernoulli beam equation

$$EI\frac{\partial^4 u}{\partial x^4} + \rho\frac{\partial^2 u}{\partial t^2} = F(x,t) \tag{1}$$

where $u(x,t)$ is the displacement of the beam's centerline from the x-axis at time t, F represents the distributed body forces, E is Young's modulus, I is moment of inertia (EI is sometimes denoted *flexural rigidity*), and ρ is (linear) density of the beam. Here we take E, I, ρ as positive constants, and define $\alpha^2 := EI/\rho$.

As a fourth-order equation we need to impose four boundary conditions. Without loss of generality, consider what boundary conditions we can impose at $x = 0$ (see figure 1):

- **free end**: $u_{xx}(0,t) = 0$, $u_{xxx}(0,t) = 0$;
- **clamped end**: $u(0,t) = 0 = u_x(0,t)$;
- **simply-supported/hinged end**: $u(0,t) = u_{xx}(0,t) = 0$.

20.1 Examples of unforced beam problems

Example: both ends simply supported (no body forces; see figure 2(a)):

$$\begin{cases} u_{tt} = -\alpha^2 u_{xxxx} & 0 < x < 1\ ,\ t > 0 \\ u(0,t) = 0 = u_{xx}(0,t) & t > 0 \\ u(1,t) = 0 = u_{xx}(1,t) & \\ u(x,0) = f(x)\ ,\ u_t(x,0) = g(x) & 0 < x < 1 \end{cases}$$

Separating variables, let $u(x,t) = X(x)T(t)$. Then

$$-\frac{1}{\alpha^2 T}\frac{d^2T}{dt^2} = \frac{1}{X}\frac{d^4X}{dx^4} = \lambda \quad \Rightarrow \quad \frac{d^2T}{dt^2} + \alpha^2\lambda T = 0\ ,$$

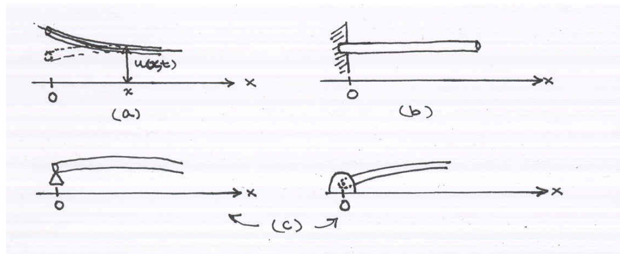

Figure 1: Visualizing beam boundary conditions at $x = 0$:(a) free-end b.c. (b) clamped-end b.c. (c) simply-supported/hinged-end b.c.

and

$$\frac{d^4X}{dx^4} - \lambda X = 0 \quad 0 < x < 1$$
$$X(0) = X(1) = \frac{d^2X}{dx^2}(0) = \frac{d^2X}{dx^2}(1) = 0 \ . \tag{2}$$

Without going into a detailed argument, assume the eigenvalues of this problem are real. Let a solution to (2) be $\{\lambda, X(x)\}$. Multiply the equation (2) by X and integrate. Since

$$\int_0^1 X\frac{d^4X}{dx^4}dx = X\frac{d^3X}{dx^3}|_0^1 - \frac{dX}{dx}\frac{d^2X}{dx^2}|_0^1 + \int_0^1 (\frac{d^2X}{dx^2})^2 dx = \int_0^1 (\frac{d^2X}{dx^2})^2 dx \ ,$$

then

$$\int_0^1 (\frac{d^2X}{dx^2})^2 dx - \lambda\int_0^1 X^2 dx = 0 \quad \Rightarrow \quad \lambda = \int_0^1 (\frac{d^2X}{dx^2})^2 dx / \int_0^1 X^2 dx \ \ .$$

Thus, $\lambda \geq 0$. If $\lambda = 0$, then $X(x) = Ax^3 + Bx^2 + Cx + D$, so $X'' = 6Ax + 2B$. $X(0) = D = 0$, $X''(0) = 2B = 0$, $X''(1) = 6A = 0$, $X(1) = C = 0$, so $X \equiv 0$. Hence, $\lambda = 0$ can not be an eigenvalue. For $\lambda > 0$, $T(t) = a\cos(\alpha\sqrt{\lambda}t) + b\sin(\alpha\sqrt{\lambda}t)$, and for the EVP, let $X = e^{rx}$; upon substituting into the equation, $r^4 - \lambda = 0$, or $r^2 = \pm\sqrt{\lambda}$, or $r = \pm\lambda^{1/4}, \pm i\lambda^{1/4}$. For convenience, let $\mu = \lambda^{1/4}$. Then $X(x) = A\cos(\mu x) + B\sin(\mu x) + Ccosh(\mu x) + Dsinh(\mu x)$. Now $X(0) = A + C = 0$, $0 = X''(0) = -\mu^2 A + \mu^2 C = -2\mu^2 A$, so $A = C = 0$. Also, $X(1) = 0 = X''(1)$ implies $D = 0$, so finally we have $B\sin(\mu) = 0$. Since $B \neq 0$ (otherwise $X \equiv 0$), so $\sin(\mu) = 0$. This gives $\mu = \mu_n = n\pi$, for $n = 1, 2, \ldots$ (so $\lambda_n^{1/2} = \mu_n^2 = n^2\pi^2$). Hence, $X(x) =$

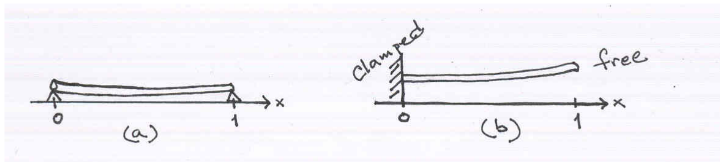

Figure 2: Beam boundary conditions for examples/exercises:(a) simply-supported b.c.s at both ends; (b) beam clamped at the $x = 0$ end, free at the $x = 1$ end

$X_n(x) = \sin(n\pi x) \Rightarrow$

$$u(x,t) = \sum_{n=1}^{\infty} \{a_n \cos(\alpha n^2\pi^2 t) + b_n \sin(\alpha n^2 \pi^2 t)\} \sin(n\pi x) \ .$$

Also,

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} a_n \sin(n\pi x) \quad \rightarrow \quad a_n = 2\int_0^1 f(x)\sin(n\pi x)dx$$

$$g(x) = u_t(x,0) = \sum_{n=1}^{\infty} \alpha n^2\pi^2 b_n \sin(n\pi x) \quad \rightarrow \quad b_n = \frac{2}{\alpha n^2\pi^2}\int_0^1 g(x)\sin(n\pi x)dx \ .$$

Exercise: For the above problem find $u(x,t)$ when
a) $f(x) = g(x) = \sin(\pi x)$;
b) $f(x) = 1 - x^2$, $g(x) \equiv 0$.

Exercise: Suppose the above beam is "weakly" damped, that is, it is still simply supported on both ends (see figure 2(a)), but now

$$u_{tt} + ku_t + \alpha^2 u_{xxxx} = 0 \ .$$

Assume $0 < k < 2\alpha\pi^2$, and show that

$$u(x,t) = e^{-kt/2} \sum_{n=1}^{\infty} \{a_n \cos(\omega_n t) + b_n \sin(\omega_n t)\} \sin(n\pi x) \ ,$$

where $\omega_n := \sqrt{4\alpha^2\lambda_n - k^2}/2$.

Exercise: Clamped end on the left, free end on the right (see figure 2(b)):

$$\begin{cases} u_{tt} = -\alpha^2 u_{xxxx} & 0 < x < 1\ ,\ t > 0 \\ u(0,t) = 0 = u_x(0,t) & t > 0 \\ u_{xx}(1,t) = 0 = u_{xxx}(1,t) & \\ u(x,0) = f(x)\ ,\ u_t(x,0) = g(x) & 0 < x < 1 \end{cases}$$

1. Obtain the fourth-order EVP and show that $\lambda = 0$ can not be an eigenvalue.

2. Obtain the transcendental relation that $\mu = \lambda^{1/4}$ satisfies, and show there is an infinite number of solutions, $0 < \mu_1 < \mu_2 < \ldots$. Thus, the associated eigenfunctions have the form

$$X_n(x) = \cos(\mu_n x) - cosh(\mu_n x) + \frac{\sin(\mu_n) - sinh(\mu_n)}{\cos(\mu_n) + cosh(\mu_n)}(\sin(\mu_n x) - sinh(\mu_n x))\ .$$

3. Assume $\{X_n\}$ are orthogonal set of functions. Write the form of $u(x,t)$ and formulas for the coefficients.

Exercise: Consider a model for a thin beam that has length 1, is clamped at $x = 0$, and is simply supported at $x = 1$. Assume general initial conditions. Find the transcendental relation that determines the eigenvalues and write out the associated eigenfunctions (no arbitrary constants). Then solve the t equation and write the series solution for the displacement u.

Exercise: Consider a flexible beam of unit length clamped at both ends. Hence, small transverse wave motion in the beam can be modeled by

$$\begin{cases} u_{tt} + \alpha^2 u_{xxxx} = 0 & 0 < x < 1\ ,\ t > 0 \\ u(0,t) = 0 = u(0,t) & t > 0 \\ u_x(1,t) = 0 = u_x(1,t) & \\ u(x,0) = f(x)\ ,\ u_t(x,0) = g(x) & 0 < x < 1\ . \end{cases}$$

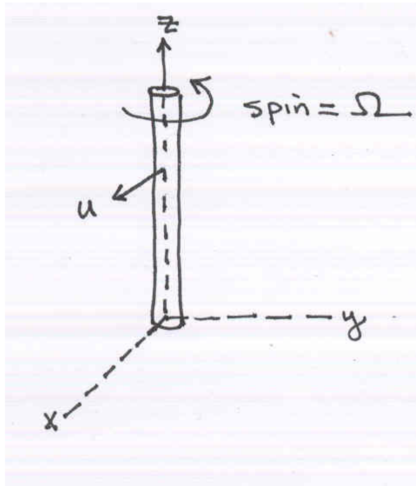

Figure 3: Rotating beam for the exercise below

Given a solution $u(x,t)$, show that the energy functional

$$\mathcal{E}(t) := \frac{1}{2}\int_0^1 \{(u_t)^2 + \alpha^2 (u_{xx})^2\} dx$$

is conserved; that is, $\mathcal{E}$ is independent of t for all $t \geq 0$.

Exercise: Consider a special case of a rotor, i.e. an Euler-Bernoulli beam that is rotating. If somehow we can have it vibrate only in one plane and be simply supported at both ends, then we might consider a model of the form

$$\begin{cases} u_{tt} + \alpha^2 u_{zzzz} - \Omega^2 u = 0 & 0 < z < l \ , \ t > 0 \\ u(0,t) = 0 = u_{zz}(0,t) & t > 0 \\ u(l,t) = 0 = u_{zz}(l,t) \ . \end{cases}$$

This is similar to the first example except for the Ω-spin term. How does this affect the vibrating modes (eigenvalues) of the problem, and what might you surmise about adding spin to the beam this way?

Remark: In thinking about the model for a piano string almost everyone models it as an ideal string

$$\rho A \frac{\partial^2 u}{\partial t^2} = T \frac{\partial^2 u}{\partial x^2} \quad 0 < x < l \ ,$$

where, as usual, T is the tension, ρ is the linear density, and A is the (uniform) cross-sectional area of the string. Howison [1] commented that a real piano string has a small bending stiffness, so a combination of the string model and the beam model might be a more appropriate model for piano wire displacement:

$$\rho A \frac{\partial^2 u}{\partial t^2} - T \frac{\partial^2 u}{\partial x^2} + EAk^2 \frac{\partial^4 u}{\partial x^4} = 0 \ ,$$

where E is Young's modulus [2] and k is radius of gyration of the cross-section. (For example, for a cross-section that is circular with radius a, $k^2 = a^2/2$.)

To get a better idea of the relative size of this new term, we need to *non-dimensionalize* (that is, *scale*) the equation. Let $\tilde{x} = x/l$, so $\partial/\partial x = (1/l)\partial/\partial\tilde{x}$, etc. by the chain rule. So we are scaling the spatial variable by the length of the string. Similarly, let $\tilde{t} = ct/l$, where $c^2 = T/\rho A$, so that c has units of length/time, i.e. of speed. Thus $\tilde{x}$ and $\tilde{t}$ are dimensionless. Now let $\tilde{u}(\tilde{x}, \tilde{t}) = u(x,t)$, substitute the derivatives into the above equation, then multiply through by $l^2/\rho A c^2$ and drop the tilde notation (for convenience). Thus,

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + \varepsilon \frac{\partial^4 u}{\partial x^4} = 0 \ \text{ where } \ \varepsilon := \frac{EAk^2}{Tl^2} \ .$$

Example: Typical values would be $a = 1\ mm$, $E \approx 2 \times 10^{11}$ (SI units), $\rho = 7800$, $l = 1\ m$, $T = 1000\ N$. This gives $\varepsilon \approx 3.1 \times 10^{-4}$, quite small, but the term is negligible and can be dropped only if the fourth-order term remains small over the whole domain, except perhaps right at the spatial boundary.

20.2 Periodic forcing of a thin beam

Historical note: Broughton bridge (Manchester, England), 1831
This suspension bridge collapsed when soldiers were marching across it. The

[1] S. Howison, **Practical Applied Mathematics: Modeling, Analysis, Approximation**, Cambridge Univ. Press, 2005

[2] Young's modulus, sometimes called the elastic modulus, is a measure of stress/strain.

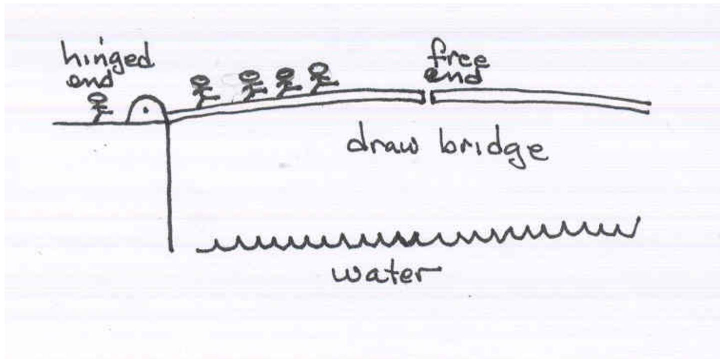

Figure 4: Cartoon of my Florida draw bridge for the periodically forced beam problem.

collapse could have been caused either by the weight of the soldiers exceeding the capacity of the bridge, or their synchronized steps causing a beating or resonance situation associated with the bridge. But, as a consequence of the incident, all armies break cadence crossing bridges. However, this has not been followed necessarily by marching bands. The author experienced a beating situation with a steel draw bridge marching across it with a band in the Orange Bowl Parade in Miami, Florida when he was in high school (see figure 4). We basically fell all over each other. The experience motivates the next problem.

Example: Consider the problem

$$\rho u_{tt} = -EIu_{xxxx} + a\cos(\omega t)\phi_m(x) \quad 0 < x < l\ ,\ t > 0 \tag{3}$$

where $\phi_m(x)$ is one of the eigenfunctions of the problem's EVP to be determined. Also

$$\begin{array}{ll} u(x,0) = 0 = u_t(x,0) & 0 < x < l \\ u(0,t) = 0 = u_{xx}(0,t) & t > 0 \\ u_{xx}(l,t) = 0 = u_{xxx}(l,t) & t > 0 \end{array} \tag{4}$$

Exercise: Show that the non-forced, **steady state** solution is $u(x) = Ax$, where A is an arbitrary constant.

Homogeneous problem:

$$\begin{cases} \rho u_{tt} = -EIu_{xxxx} & 0 < x < l \\ \\ u(0,t) = 0 = u_{xx}(0,t) & t > 0 \\ \\ u_{xx}(l,t) = 0 = u_{xxx}(l,t) \ . \end{cases}$$

With $u(x,t) = T(t)\phi(x)$, we have $\frac{\rho}{EI}\frac{1}{T}\frac{d^2T}{dt^2} = -\frac{1}{\phi}\frac{d^4\phi}{dx^4} = -\lambda$, so

$$\begin{cases} \frac{d^4\phi}{dx^4} - \lambda\phi = 0 & 0 < x < l \\ \\ \phi(0) = 0 = \frac{d^2\phi}{dx^2}(0) \\ \\ \frac{d^2\phi}{dx^2}(l) = 0 = \frac{d^3\phi}{dx^3}(l) \end{cases}$$

Assume $\lambda \geq 0$. For the characteristic equation, $\phi = e^{rx}$, we obtain $r^4 - \lambda$, or $r^2 = \pm\sqrt{\lambda}$, which gives $r = \pm\lambda^{1/4},\ \pm i\lambda^{1/4}$, assuming $\lambda > 0$. If $\lambda = 0$, from the exercise above, $\phi = \phi_0(x) = Ax$, that is, 0 is an eigenvalue. Now for $\lambda > 0$, define $\mu := \lambda^{1/4}$. Then $\phi(x) = Acosh(\mu x) + Bsinh(\mu x) + C\cos(\mu x) + D\sin(\mu x)$. Also, $\phi''(x) = \mu^2\{Acosh(\mu x) + Bsinh(\mu x) - C\cos(\mu x) - D\sin(\mu x)\}$. So $\phi(0) = A + C$ and $\phi''(0) = \mu^2(A - C)$. Hence, $A = C = 0$, so $\phi''(x) = \mu^2\{Bsinh(\mu x) - D\sin(\mu x)\}$ and $\phi'''(x) = \mu^3\{Bcosh(\mu x) - D\cos(\mu x)\}$. Thus

$$\begin{cases} \phi''(l) = 0 = \mu^2\{Bsinh(\mu l) - D\sin(\mu l)\} \\ \phi'''(l) = 0 = \mu^3\{Bcosh(\mu l) - D\cos(\mu l)\} \end{cases} .$$

This can be written in matrix form as

$$\begin{bmatrix} sinh(\mu l) & -\sin(\mu l) \\ \\ cosh(\mu l) & -\cos(\mu l) \end{bmatrix} \begin{bmatrix} B \\ \\ \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ \\ \\ 0 \end{bmatrix}$$

Since B, D can not be zero, we need the determinent of the matrix to be 0 for a non-zero solution vector. Hence, $cosh(\mu l)\sin(\mu l) - sinh(\mu l)\cos(\mu l) = 0$, or

$$tanh(\mu l) = tan(\mu l) \ .$$

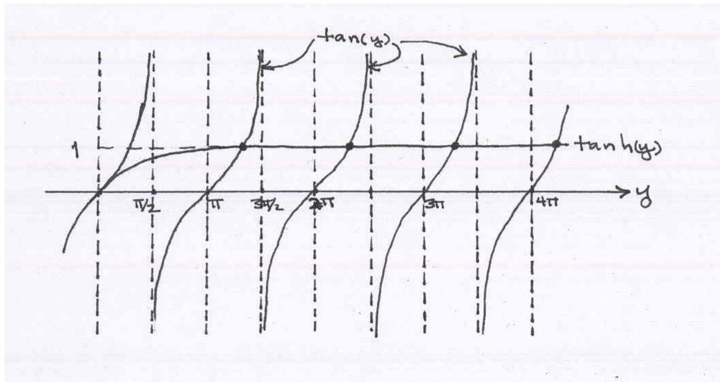

Figure 5: Sketch of graph of $tanh(y)$ versus $tan(y)$, showing the ordered sequence of positive eigenvalues.

Let $y = \mu l$, then from figure 5 we see that there is an ordered increasing set of solutions y_k, $k = 1, 2, \ldots$. For "large" k, $y_k \sim arctan(1) + k\pi = \pi/4 + k\pi$, and $y_k = \mu_k l$, or $\lambda_k = (y_k/l)^4$. Now $Bsinh(y_k) - D\sin(y_k) = 0$, so $B = B_k = \frac{\sin(y_k)}{sinh(y_k)} D_k$, which gives, up to a multiplicative constant,

$$\phi_n(x) = \sin(\mu_n x) + \frac{\sin(\mu_n l)}{sinh(\mu_n l)} sinh(\mu_n x) \quad .$$

So, in equation (3) the spatial part of the forcing is one of these eigenfunctions.

Now, for equation (3), let $u(x,t) = T(t)\phi_m(x)$; then,

$$\rho \frac{d^2 T}{dt^2}(t)\phi_m(x) = -EIT(t)\frac{d^4\phi_m}{dx^4}(x) + a\cos(\omega t)\phi_m(x) \quad ,$$

but since $d^4\phi_m/dx^4 = \lambda_m \phi_m$, then

$$\frac{d^2T}{dt^2} + \alpha^2 \lambda_m T = \frac{a}{\rho}\cos(\omega t) \qquad t > 0$$

$$T(0) = 0 = \frac{dT}{dt}(0) \quad .$$

Case 1: $\omega^2 \neq \frac{EI}{\rho}\lambda_m$
Thus, the right side function is **not** a solution to the homogeneous equation.

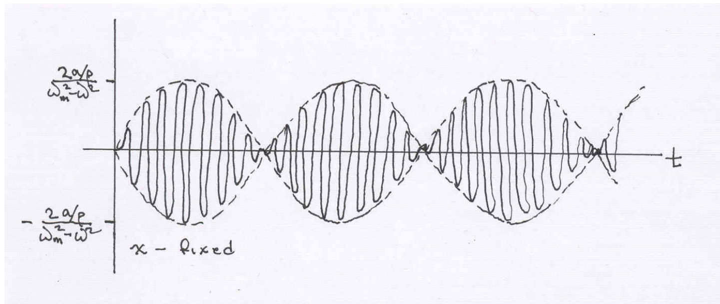

Figure 6: Beating behavior as discussed in case 1 of the periodically forced beam problem.

So let $T_{part}(t) = K\cos(\omega t)$; upon substituting into the equation, we have $-\omega^2 K + (EI/\rho)\lambda_m K = a/\rho$. Define $\omega_m^2 := \frac{EI}{\rho}\lambda_m$. Then $K = \frac{a/\rho}{\omega_m^2-\omega^2}$, so

$$T(t) = A\cos(\omega_m t) + B\sin(\omega_m t) + \frac{a/\rho}{\omega_m^2-\omega^2}\cos(\omega t) \ .$$

Now $T(0) = A + \frac{a/\rho}{\omega_m^2-\omega^2} = 0$ and $\frac{dT}{dt}(0) = \omega_m B = 0$, so

$$\begin{aligned} T(t) &= \frac{a/\rho}{\omega_m^2-\omega^2}\{\cos(\omega t) - \cos(\omega_m t)\} \\ &= [\frac{2a/\rho}{\omega_m^2-\omega^2}\sin(\frac{\omega_m-\omega}{2}t)]\sin(\frac{\omega_m+\omega}{2}t) \ . \end{aligned}$$

after use of a trig addition formula. In the interesting case where $0 < |\omega_m - \omega| << 1$, we have a large amplitude factor multiplying a "large" frequency $\sin(\frac{\omega_m+\omega}{2}t)$, namely the term in square brackets, with this amplitude having "small" frequency = large period (see figure 6).

Case 2: $\omega^2 = \omega_m^2$
In this case the forcing function is a solution to the homogeneous equation, so

$$\frac{d^2T}{dt^2} + \omega^2 T = \frac{a}{\rho}\cos(\omega t)$$

If we let $T_{part}(t) = K_1 t\cos(\omega t) + K_2 t\sin(\omega t)$ via the undetermined coefficient method, then upon substituting this into the equation, and using the initial conditions, we obtain $K_1 = 0$, and

$$T(t) = \frac{at}{2\rho\omega} \sin(\omega t) \ .$$

That is, we have the **resonance** condition of having the amplitude go unbounded as $t \to \infty$. Finally, in this case $u(x,t) = \frac{at}{2\rho\omega} \sin(\omega t)\phi_m(x)$. (This case is not realistic since the model assumptions do not allow for large amplitude motion.)